J. Evol. Equ. 21 (2021), 3675–3701

1424-3199/21/033675-27, *published online* May 25, 2021
https://doi.org/10.1007/s00028-021-00691-5

Journal of Evolution Equations

Nonexistence results for parabolic equations involving the *p*-Laplacian and Hardy–Leray-type inequalities on Riemannian manifolds

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Dedicated to Matthias Hieber on the occasion of his 60th birthday

Abstract. The main goal of this paper is twofold. The first one is to investigate the nonexistence of positive solutions for the following nonlinear parabolic partial differential equation on a noncompact Riemannian manifold M,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{p,g} u + V(x)u^{p-1} + \lambda u^q & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $1 < p < 2$, $V \in L^1_{\text{loc}}(\Omega)$, $q > 0$, $\lambda \in \mathbb{R}$, Ω is bounded and has a smooth boundary in M and $\Delta_{p,g}$ is the p-Laplacian on M. The second one is to obtain Hardy- and Leray-type inequalities with remainder terms on a Riemannian manifold M that provide us concrete potentials to use in the partial differential equation we are interested in. In particular, we obtain explicit (mostly sharp) constants for these inequalities on the hyperbolic space $\mathbb{H}^n$.

1. Introduction

Let M be an n-dimensional complete noncompact Riemannian manifold, $n \geq 2$, endowed with a Riemannian metric tensor $g = (g_{ij})$, and Ω be a bounded domain with smooth boundary in M. One of the main goals of this paper is to investigate nonexistence of positive solutions of the following non-Newtonian filtration equation with reaction sources and potential,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{p,g} u + V(x)u^{p-1} + \lambda u^q & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $1 < p < 2$, $q > 0$, $\lambda \in \mathbb{R}$, $V \in L^1_{loc}(\Omega)$ and $\Delta_{p,g} u = -\text{div}_g(|\nabla_g u|^{p-2}\nabla_g u)$ is the p-Laplacian with $1 < p < \infty$.

Keywords: Critical exponent, Hardy–Leray-type inequalities, Nonexistence of positive solutions.

The class (1.1) of nonlinear partial differential equations includes several important cases. Before proceeding to our own results, let us briefly review some important developments that motivated our study. We begin by recalling the results from the linear case.

Linear problems. Let $M = \mathbb{R}^n$ with the Euclidean metric tensor $g_{ij} = \delta_{ij}$. If $p = 2$ and $\lambda = 0$, then our model problem (1.1) reduces to linear heat problem with potential

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + V(x)u & \text{in} \quad \Omega \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{1.2}$$

If the potential V belongs to the Kato class or L^p with $p > n/2$ then the Hamiltonian $\mathcal{H} = -\Delta - V$ on $L^2(M)$ has several good properties and so the linear heat problem (1.2) is well understood. If the potential V does not belong to these classes, such as $V = c/|x|^2$, then the solutions of heat problem may have critical behavior. An interesting result in this direction was obtained by Baras and Goldstein [5]. They showed that the following heat problem,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2} u & \text{in} \quad \Omega \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x, t) = 0 & \text{on} \quad \partial\Omega \times (0, T), \end{cases} \tag{1.3}$$

has no nonnegative solutions except $u \equiv 0$ if $c > \mathrm{C_H} = (\frac{n-2}{2})^2$, and positive weak solutions do exist if $c \leq \mathrm{C_H}$. Thus, $\mathrm{C_H} = (\frac{n-2}{2})^2$ is the cut-off point for existence of positive solutions for the heat equation with inverse square potential $c/|x|^2$. The critical constant $\mathrm{C_H}$ is the best constant in Hardy's inequality,

$$\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx,$$

valid for all $\phi \in C_c^\infty(\mathbb{R}^n)$ if $n \geq 3$ and all $\phi \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$ if $n = 1, 2$.

In an interesting paper, Cabré and Martel [8] extended some of the results of Baras and Goldstein [5] to general positive singular potentials. They discovered that existence and nonexistence of positive solutions of the problem (1.2) is largely determined by the size of the infimum of the spectrum of the symmetric operator $\mathcal{H} = -\Delta - V$, which is

$$\sigma_{\inf}(V; \Omega) := \inf_{0 \not\equiv \phi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\phi|^2 dx - \int_\Omega V|\phi|^2 dx}{\int_\Omega |\phi|^2 dx}. \tag{1.4}$$

The observation of Cabré and Martel [8] was used in [16,17,19,22] to get additional results on nonexistence of positive solutions for wide classes of linear parabolic problems.

Nonlinear problems. Let $M = \mathbb{R}^n$ with the Euclidean metric tensor $g_{ij} = \delta_{ij}$. In his pioneering paper [14], Fujita studied the following Cauchy problem for the semilinear heat equation

$$\begin{cases} u_t = \Delta u + u^p, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $p > 1$ and $u_0(x)$ is a bounded positive continuous function. He proved that

(i) if $1 < p < p_F$ then (1.5) has no global positive solutions;

(ii) if $p > p_F$ and $u_0 \leq \delta e^{-|x|^2}$ $(0 < \delta \ll 1)$, then (1.5) has global positive solutions,

where $p_F = 1 + \frac{2}{n}$. We call this critical number $p_F = 1 + \frac{2}{n}$ the Fujita exponent. The statement (i) also holds for the critical case $p = p_F$, which was proved later by Hayakawa [20] and Weissler [37]. The result of Fujita [14] has been extended and generalized to various directions. For instance, the case of a domain, bounded or unbounded, replacing $\mathbb{R}^n$ in (1.5), has been considered. Furthermore, different equations have been studied, involving more general reaction terms instead of u^p or non-parabolic operators instead of the heat operator $\partial_t - \Delta$. We refer the reader to the survey papers by Deng and Levine [12] and Levine [28] for a good account of related works.

At the same time, extensions have been carried out also in the context of Riemannian manifolds. In a series of articles [38–41], not only did Zhang generalize the result of Fujita to the case of Riemannian manifolds, but he also extended it for wide class of nonlinear parabolic problems on Riemannian manifolds. For instance, Zhang [38] considered the following semilinear heat equation with a potential term on an n $(n \geq 3)$-dimensional complete noncompact Riemannian manifold M,

$$\begin{cases} u_t = \Delta_g u - V(x)u + u^q & \text{in} \quad M \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad M, \end{cases} \tag{1.6}$$

where $q > 1$. He studied the relation between the Fujita exponent and the potentials V behaving like $\frac{a}{(1+d(x)^b)}$, by using global bounds for the fundamental solutions of the heat equations with a potential. Here, $a \in \mathbb{R}$, $b > 0$, and $d(x)$ is the distance between a point $x \in M$ and a reference point $O \in M$. Indeed, Zhang's result is a concrete example of how the potential V has a strong influence on the Fujita exponent.

When $M = \mathbb{R}^n$ with the Euclidean metric tensor $g_{ij} = \delta_{ij}$ and $\lambda = 0$, then problem (1.1) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_p u + V(x)u^{p-1} & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases} \tag{1.7}$$

Using the method of Cabré and Martel [8], Goldstein and Kombe [18] showed that nonexistence of positive solutions of the problem (1.7) is determined by the value of p and the size of the following normalized p-energy form

$$\sigma_{\inf}^p := \inf_{0 \not\equiv \phi \in C_c^\infty(\Omega \setminus \mathcal{K})} \frac{\int_\Omega |\nabla\phi|^p dx - \int_\Omega V|\phi|^p dx}{\int_\Omega |\phi|^p dx}, \tag{1.8}$$

where $\mathcal{K}$ is a closed Lebesgue null subset of Ω.

In light of these developments, it is natural to ask how the addition of the source term λu^q to the problem (1.7) will affect the results previously obtained in [18]. The first purpose of this article is to address the question above in an n-dimensional complete noncompact Riemannian manifold M.

Note that there is a competition between the integral terms $\int_\Omega |\nabla \phi|^p dv_g$ and $\int_\Omega V|\phi|^p dv_g$ in the the bottom of the spectrum (1.8), and one could expect that the the bottom of the spectrum (1.8) can be $-\infty$. In fact, this depends on the choice of the potential V. Our interest here is to consider only the critical potentials, which are related to sharp Hardy and Leray type inequalities. Our second main goal in this paper is to present various Hardy- and Leray-type inequalities with remainder terms on a Riemannian manifold M. In particular, we obtain sharp constants for these inequalities on the hyperbolic space $\mathbb{H}^n$.

The plan of this paper is as follows. In Sect. 2, some preliminaries are introduced and one of the main results of this paper is proved. Section 3 is devoted to the study of Hardy and Leray type inequalities with remainders. Furthermore, we consider the hyperbolic space $\mathbb{H}^n$ as a model manifold and recall some facts about the hyperbolic space. Then, we present various Hardy- and Leray-type inequalities with explicit constants (mostly sharp) and then the application of Theorem 2.1.

2. Preliminaries and nonexistence of positive solutions

In this section, we will recall some basic definitions, notations and results used in this paper. We refer to [10,15] for more precise information about this subject.

Let $M \equiv (M, g)$ be an n-dimensional complete Riemannian manifold endowed with a metric tensor $g = (g_{ij})$. In local coordinates $x = (x_1, x_2, \ldots, x_n)$, the Riemannian metric is written in the form

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j,$$

where $(g_{ij})_{i,j=1}^n$ is a symmetric positive definite matrix of smooth functions.

The volume element of M is given in the same local coordinates by

$$dv_g = \sqrt{\det(g_{ij})} dx_1 dx_2 \ldots dx_n,$$

and the volume of a bounded open set $S \subset M$ is

$$\mathrm{Vol}_g(S) = \int_S dv_g.$$

Let $u : M \to \mathbb{R}$ be a given smooth function. The Riemannian gradient of u is the vector field $\nabla_g u$ defined by

$$\langle \nabla_g u, X \rangle = du(X) = Xu$$

for all smooth vector fields X on M. Here, $\langle\cdot,\cdot\rangle$ is the scalar product induced by g. In local coordinates, we have

$$\nabla_g u = \Big((\nabla_g u)_1, \ldots, (\nabla_g u)_n\Big) \quad \text{with} \quad (\nabla_g u)_i := \sum_{j=1}^n g^{ij} \frac{\partial u}{\partial x_j}$$

and

$$|\nabla_g u|^2 = \langle \nabla_g u, \nabla_g u\rangle = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j},$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

The divergence $\mathrm{div}_g X$ of a vector field X is defined as the unique smooth function on M such that

$$\int_M f \mathrm{div}_g X dv_g = -\int_M \langle \nabla f, X\rangle dv_g,$$

for all $f_c^\infty(M)$. In local coordinates, if $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ then

$$\mathrm{div}_g X = \frac{1}{\sqrt{det(g_{ij})}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\sqrt{det(g_{ij})} X_i\Big).$$

The Laplace–Beltrami operator in M of a function $u \in C^2(M)$ is given by

$$\Delta_g u = \mathrm{div}_g(\nabla_g u)$$

and whose expression in local coordinates is

$$\Delta_g u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\sqrt{\det(g_{ij})} \sum_{j=1}^n g^{ij} \frac{\partial u}{\partial x_j}\Big).$$

Due to the divergence theorem, we have

$$\int_M \phi \Delta_g u dv_g = -\int_M \langle \nabla_g \phi, \nabla_g u\rangle dv_g,$$

for any smooth $u, \phi : M \to \mathbb{R}$, with either u or ϕ compactly supported.

The p-Laplace operator $\Delta_{p,g}$ acts on C^3 functions u on M and is given by

$$\Delta_{p,g} u = \mathrm{div}_g(|\nabla_g u|^{p-2} \nabla_g u),$$

where $1 < p < \infty$. Note that if $p = 2$, the p-Laplace operator is nothing but the Laplace–Beltrami operator.

Sobolev inequality. A Riemannian Manifold (M, g) of dimension $\dim M = n > p \geq 1$ is said to support a Euclidean-type L^p Sobolev inequality if there exists a constant $\mathrm{C}_M > 0$ such that, for every $\phi \in C_c^\infty(M)$,

$$\Big(\int_M |\phi|^{p^*} dv_g\Big)^{\frac{1}{p^*}} \leq \mathrm{C}_M \Big(\int_M |\nabla_g \phi|^p dv_g\Big)^{\frac{1}{p}} \tag{2.1}$$

where

$$p^* = \frac{np}{n-p}.$$

It is well-known that the Sobolev inequality (2.1) holds in $\mathbb{R}^n$ and plays a key role in analysis in Euclidean spaces and in the study of solutions of partial differential equations. However, for a general manifold, it may not be true. We refer to Saloff-Coste [34] and Hebey [21] for discussions of the validity of Sobolev inequalities on a manifold.

As a result of Hölder- and Sobolev-type inequalities, we give the following lemma which is a key one in order to establish our first main result.

Lemma 2.1. *Let M be a complete noncompact Riemann manifold of dimension $n \geq 2$, $1 < p < n$ and Ω be a bounded domain with smooth boundary in M. Assume that $a(x) \in L^{\frac{n}{p}}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. Then, for each $\epsilon > 0$ there exist a positive constant $C(\epsilon)$ such that*

$$\left|\int_\Omega a(x)|\phi|^p dv_g\right| \leq \frac{\epsilon}{2(1-\epsilon)}\int_\Omega |\nabla_g\phi|^p dv_g + C(\epsilon)\int_\Omega |\phi|^p dv_g. \tag{2.2}$$

Proof. Let $(a_k)_{k\geq 1}$ be the sequence defined by $a_k(x) = \min\{a(x), k\}$ for almost every $x \in \Omega$ and $k \geq 1$. Then it can be shown that

$$||a_k(x) - a(x)||_{L^{n/p}(\Omega)} \longrightarrow 0 \quad as \quad k \longrightarrow \infty. \tag{2.3}$$

Clearly we have,

$$\left|\int_\Omega a(x)|\phi|^p dv_g\right| \leq \int_\Omega |a - a_k||\phi|^p dv_g + k\int_\Omega |\phi|^p dv_g. \tag{2.4}$$

By Hölder's inequality with conjugate exponents $\frac{n}{n-p}$ and $\frac{n}{p}$, we get

$$\left|\int_\Omega a|\phi|^p dv_g\right| \leq \Big(\int_\Omega |a-a_k|^{\frac{n}{p}} dv_g\Big)^{\frac{p}{n}}\Big(\int_\Omega |\phi|^{\frac{np}{n-p}} dv_g\Big)^{\frac{n-p}{n}} + k\int_\Omega |\phi|^p dv_g. \tag{2.5}$$

We remark that Sobolev inequality (2.1) always holds since Ω is a bounded domain (see [34]). Hence,

$$\left|\int_\Omega a|\phi|^p dv_g\right| \leq \Big(\int_\Omega |a - a_k|^{\frac{n}{p}} dv_g\Big)^{\frac{p}{n}}\left(\mathrm{C}_M^p\int_\Omega |\nabla_g\phi|^p dv_g\right) + k\int_\Omega |\phi|^p dv_g. \tag{2.6}$$

Due to the limit (2.3), for every given $\epsilon \in (0, 1)$, there is an $k(\epsilon) \geq 1$ such that

$$\Big(\int_\Omega |a - a_k|^{\frac{n}{p}} dv_g\Big)^{\frac{p}{n}} \leq \frac{\epsilon}{2(1-\epsilon)\mathrm{C}_M^p}. \tag{2.7}$$

Let $k = C(\epsilon)$; we have the desired inequality (2.2). $\square$

Before we present our main results, we define the generalized positive local solution in the following sense.

Definition. By a *positive local solution of* (1.1) *continuous* off of $\mathcal{K}$, we mean

(i) $\mathcal{K}$ is a closed Lebesgue null subset of Ω,
(ii) $u : [0, T) \longrightarrow L^1(\Omega)$ is continuous for some $T > 0$,
(iii) $(x, t) \mapsto u(x, t) \in C((\Omega\backslash\mathcal{K}) \times (0, T))$,
(iv) $u(x, t) > 0$ on $(\Omega\backslash\mathcal{K}) \times (0, T)$,
(v) $\lim_{t\to 0} u(., t) = u_0$ in the sense of distributions,
(vi) $\nabla_g u \in L^p_{\text{loc}}(\Omega)$ and u is a solution in the sense of distributions of the PDE, in $\Omega \times (0, T)$.

Remark 2.1. If $0 < a < b < T$ and $\mathcal{K}_o$ is a compact subset of $\Omega\backslash\mathcal{K}$, then $u(x, t) \geq \epsilon_1 > 0$ for $(x, t) \in \mathcal{K}_o \times [a, b]$ for some $\epsilon_1 > 0$. We can weaken (iii), (iv) to be

(iii)' $u(x, t)$ is positive and locally bounded on $(\Omega\backslash\mathcal{K}) \times (0, T)$,
(iv)' $\frac{1}{u(x,t)}$ is locally bounded on $(\Omega\backslash\mathcal{K}) \times (0, T)$.

If a solution satisfies (i), (ii), (iii)', (iv)', (v), and (vi), then we call it a *generalized positive local solution off of* $\mathcal{K}$. This is more general than a positive local solution continuous off of $\mathcal{K}$. If $\mathcal{K} = \emptyset$, we simply call u *generalized positive local solution*.

Now, we are able to give our first main result of this section.

Theorem 2.1. *Let M and Ω be as above, $n \geq 2$, $\frac{2n}{n+1} \leq p < 2$, $p-1 < q < p+\frac{p-n}{n}$, $V(x) \in L^1_{loc}(\Omega\backslash\mathcal{K})$ where $\mathcal{K}$ is a Lebesgue null subset of Ω. If*

$$\sigma^p_{\text{inf}} := \inf_{0\not\equiv\phi\in C_c^\infty(\Omega\backslash\mathcal{K})} \frac{\int_\Omega |\nabla_g\phi|^p dv_g - \int_\Omega (1-\epsilon)V|\phi|^p dv_g}{\int_\Omega |\phi|^p dv_g} = -\infty \tag{2.8}$$

for some $\epsilon > 0$, then the problem (1.1) has no generalized positive local solution off of $\mathcal{K}$.

Proof. We argue by contradiction. Given any $T > 0$, let $u : [0, T) \longrightarrow L^1(\Omega)$ be a generalized positive local solution to (1.1) in $(\Omega\backslash\mathcal{K}) \times (0, T)$ with $u_0 \geq 0$ but not identically zero.

We multiply both sides of the first equation in (1.1) by the test function $|\phi|^p/u^{p-1}$, where $\phi \in C_c^\infty(\Omega\backslash\mathcal{K})$ and integrate over Ω. The result is

$$\begin{aligned}\frac{1}{2-p}\frac{d}{dt}\int_\Omega u^{2-p}|\phi|^p dv_g &= \int_\Omega \operatorname{div}_g\left(|\nabla_g u|^{p-2}\nabla_g u\right)\frac{|\phi|^p}{u^{p-1}}dv_g\\ &\quad+\int_\Omega V(x)|\phi|^p dv_g + \int_\Omega \lambda u^{q-p+1}|\phi|^p dv_g.\end{aligned} \tag{2.9}$$

The divergence theorem gives

$$A = \int_\Omega \operatorname{div}_g\left(|\nabla_g u|^{p-2}\nabla_g u\right)\frac{|\phi|^p}{u^{p-1}}dv_g = -\int_\Omega |\nabla_g u|^{p-2}\left\langle\nabla_g u, \nabla_g\left(\frac{|\phi|^p}{u^{p-1}}\right)\right\rangle dv_g. \tag{2.10}$$

Direct computation shows that

$$-|\nabla_g u|^{p-2}\left\langle \nabla_g u, \nabla_g \left(\frac{|\phi|^p}{u^{p-1}}\right)\right\rangle = -\, p|\nabla_g u|^{p-2}\frac{|\phi|^{p-1}}{u^{p-1}}\langle \nabla_g u, \nabla_g|\phi|\rangle \\ + (p-1)\frac{|\phi|^p}{u^p}|\nabla_g u|^p. \tag{2.11}$$

Hence,

$$A = \int_\Omega \operatorname{div}_g \left(|\nabla_g u|^{p-2}\nabla_g u\right) \frac{|\phi|^p}{u^{p-1}} dv_g \geq B \tag{2.12}$$

where

$$B = \int_\Omega \left((p-1)|\nabla_g u|^p \frac{|\phi|^p}{u^p} - p|\nabla_g u|^{p-1}\frac{|\phi|^{p-1}}{u^{p-1}}|\nabla_g \phi|\right) dv_g.$$

Here, we can use the following elementary inequality. Let $p > 1$ and $\omega_1 \neq \omega_2$ be two positive real numbers. Then,

$$\omega_1^p - \omega_2^p - p\omega_2^{p-1}(\omega_1 - \omega_2) > 0,$$

whence

$$(p-1)w_2^p - pw_2^{p-1}w_1 > -w_1^p.$$

We can take $w_2 = \frac{|\phi|}{u}|\nabla_g u|$ and $w_1 = |\nabla_g \phi|$; then we have

$$B \geq -\int_\Omega |\nabla_g \phi|^p dv_g. \tag{2.13}$$

Substituting (2.13) into (2.12) gives

$$A = \int_\Omega \operatorname{div}_g(|\nabla_g u|^{p-2}\nabla_g u)\left(\frac{|\phi|^p}{u^{p-1}}\right) dv_g \geq -\int_\Omega |\nabla_g \phi|^p dv_g. \tag{2.14}$$

Substituting (2.14) into (2.9) and integrating from t_1 to t_2, where $0 < t_1 < t_2 < T$, we obtain

$$\int_\Omega V(x)|\phi|^p dv_g - \int_\Omega |\nabla_g \phi|^p dv_g \leq L + R, \tag{2.15}$$

where

$$L = \frac{1}{(2-p)(t_2 - t_1)} \int_\Omega \left(u^{2-p}(x, t_2) - u^{2-p}(x, t_1)\right) |\phi|^p dv_g$$

and

$$R = -\lambda \int_\Omega \int_{t_1}^{t_2} u^{q-p+1}|\phi|^p dt\, dv_g.$$

Using Jensen's inequality for concave functions, we obtain, since $\frac{2n}{n+1} \leq p < 2$,

$$\int_\Omega (u(x, t_i))^{\frac{(2-p)n}{p}} dv_g \leq C(\mathrm{Vol}(\Omega))\left(\int_\Omega u(x, t_i) dv_g\right)^{\frac{(2-p)n}{p}} < \infty.$$

Therefore,

$$u^{2-p}(x,t_i) \in L^{\frac{n}{p}}(\Omega).$$

For the second integral on the right-hand side of (2.15), let $F(x) := -\lambda \int_{t_1}^{t_2} u^{q-p+1}dt$. Applying Jensen's inequality, we deduce

$$F(x) \in L^{\frac{n}{p}}(\Omega).$$

By Lemma 2.1, we have

$$\begin{aligned} R &\leq \frac{\epsilon}{2(1-\epsilon)} \int_\Omega |\nabla_g \phi|^p dv_g + C(\epsilon) \int_\Omega |\phi|^p dv_g \\ &\quad + \frac{\epsilon}{2(1-\epsilon)} \int_\Omega |\nabla_g \phi|^p dv_g + C(\epsilon) \int_\Omega |\phi|^p dv_g \\ &= \frac{\epsilon}{(1-\epsilon)} \int_\Omega |\nabla_g \phi|^p dv_g + C(\epsilon) \int_\Omega |\phi|^p dv_g. \end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.15) gives

$$\int_\Omega V(x)|\phi|^p dv_g - \int_\Omega |\nabla_g \phi|^p dv_g \leq \frac{\epsilon}{1-\epsilon} \int_\Omega |\nabla_g \phi|^p dv_g + C(\epsilon) \int_\Omega |\phi|^p dv_g. \tag{2.17}$$

Therefore,

$$\inf_{0 \not\equiv \phi \in C_c^\infty(\Omega \setminus \mathcal{K})} \frac{\int_\Omega |\nabla_g \phi|^p dv_g - \int_\Omega (1-\epsilon) V(x)|\phi|^p dv_g}{\int_\Omega |\phi|^p dv_g} \geq -(1-\epsilon)C(\epsilon) > -\infty. \tag{2.18}$$

This contradicts our assumption. The proof of the Theorem (2.1) is now complete. □

Remark 2.2. It is now clear that assuming the existence of a positive solution of problem (1.1) implies the following Hardy-type inequality,

$$\int_\Omega |\nabla_g \phi|^p dv \geq (1-\epsilon) \int_\Omega V(x)|\phi|^p dv - (1-\epsilon)C(\epsilon, a, \lambda, q, \mathrm{Vol}(\Omega)) \int_\Omega |\phi|^p dv. \tag{2.19}$$

As we mentioned above, our second goal is to present Hardy- and Leray-type inequalities with remainders on a complete noncompact Riemannian manifold M.

3. Improved Hardy- and Leray-type inequalities and applications

Hardy inequality. Let Ω be a smooth bounded domain with $0 \in \Omega$ in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$. The L^p version of Hardy's inequality states that

$$\int_\Omega |\nabla \phi(x)|^p dx \geq \left|\frac{n-p}{p}\right|^p \int_\Omega \frac{|\phi(x)|^p}{|x|^p} dx, \tag{3.1}$$

and holds for all $\phi \in C_c^\infty(\Omega)$ if $1 < p < n$, and for all $\phi \in C_c^\infty(\Omega\backslash\{0\})$ if $p \geq n$. Here, the constant $|\frac{n-p}{p}|^p$ is sharp, in the sense that

$$\inf_{0\not\equiv\phi\in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\phi|^p dx}{\int_\Omega \frac{|\phi|^p}{|x|^p} dx} = \left|\frac{n-p}{p}\right|^p .$$

The interest in the inequality $p \geq n$ is due to the fact that in (3.1) for $p = n$, $p - n = 0$ cannot be replaced by any positive number.

Leray inequality. If $n = p$, then there is a different version of the Hardy inequality. In [26], Leray proved the following integral inequality, which involves singularities at both the center and boundary of the two-dimensional unit ball,

$$\int_{B_1} |\nabla\phi|^2 dx \geq \frac{1}{4}\int_{B_1} \frac{|\phi|^2}{|x|^2\left(\ln\left(\frac{1}{|x|}\right)\right)^2} dx, \tag{3.2}$$

where B_1 is the unit ball in $\mathbb{R}^2$ centered at the origin and $\phi \in C_c^\infty(B_1\backslash\{0\})$. Furthermore, Adimurthi and Sandeep [2] obtained the multidimensional form of (3.2) and also showed that the constant $\frac{1}{4}$ is sharp,

$$\inf_{0\not\equiv\phi\in C_c^\infty(B_1\backslash\{0\})} \frac{\int_{B_1} |\nabla\phi|^2 dx}{\int_{B_1} \frac{|\phi|^2}{|x|^2\left(\ln\left(\frac{1}{|x|}\right)\right)^2} dx} = \frac{1}{4}.$$

There has been a lot of research concerning Hardy and Leray inequalities on the Euclidean space $\mathbb{R}^n$ and, in particular, sharp inequalities as well as their improved versions (in the sense that nonnegative terms are added in the right-hand side of (3.1) and (3.2)) which have attracted a lot of attention because of their application to singular problems, e.g., [1,3,4,7,13,32,36] and the references therein.

On the other hand, there has been a growing literature on Hardy and Leray inequalities in Riemannian manifolds. In an interesting paper, Carron [9] studied weighted L^2-Hardy inequalities under some geometric assumptions on the weight function ρ and obtained, among other results, the following inequality,

$$\int_M \rho^\alpha |\nabla_g\phi|^2 dx \geq \left(\frac{C+\alpha-1}{2}\right)^2 \int_M \rho^\alpha \frac{\phi^2}{\rho^2} dx, \tag{3.3}$$

where $\alpha \in \mathbb{R}$, $C+\alpha-1 > 0$, $\phi \in C_c^\infty(M\backslash\rho^{-1}\{0\})$ and the weight function ρ satisfies $|\nabla_g\rho| = 1$ and $\Delta_g\rho \geq \frac{C}{\rho}$ in the sense of distributions.

Recall that a complete Riemannian manifold M is said to be nonparabolic if it admits a (minimal) positive Green function G for Δ_g. In [27], Li and Wang proved, among other results, that existence of a weighted Hardy-type inequality is equivalent to nonparabolicity. Furthermore, they obtained the following L^2-Hardy inequality

$$\int_M |\nabla_g\phi|^2 dv \geq \frac{1}{4}\int_M \frac{|\nabla_g G(p,x)|^2}{G^2(p,x)}\phi^2 dv,$$

where $\phi \in C_c^\infty(M)$ and $G(p,x)$ is the minimal positive Green's function defined on M with a pole at the point $p \in M$.

Under the same hypotheses on the weight function ρ in [9], Kombe and Özaydın [23] extended the result of Carron to the case $p \neq 2$. In [24], they obtained the following (improved) Hardy inequality involving two weight functions ρ and δ

$$\int_M |\nabla_g \phi|^2 dv \geq \left(\frac{C-1}{2}\right)^2 \int_M \frac{\phi^2}{\rho^2} dv + \frac{1}{4}\int_M \frac{|\nabla_g \delta|^2}{\delta^2}\phi^2 dv, \tag{3.4}$$

where $\phi \in C_c^\infty(M)$, $C > 1$ and $-\mathrm{div}_g(\rho^{1-C}\nabla_g \delta) \geq 0$ in the sense of distributions. Let Ω be a bounded domain with smooth boundary in M and $\sup_\Omega(\rho) < 1$. Then, the choice of $\delta = \log(\frac{1}{\rho})$ in (3.4) yields

$$\int_\Omega |\nabla_g \phi|^2 dv \geq \frac{1}{4}\int_\Omega \frac{|\phi|^2}{\rho^2\left(\ln\left(\frac{1}{\rho}\right)\right)^2} dv, \tag{3.5}$$

which is the analogue of Leray's inequality (3.2) .

On the other hand, D'Ambrosio and Dipierro [11] obtained the following inequality which includes the Hardy- and Leray-type inequalities,

$$\int_M |\nabla_g \phi|^p dv \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla_g \rho|^p}{\rho^p}|\phi|^p dv, \tag{3.6}$$

where $\phi \in C_c^\infty(M)$, $p > 1$ and the weight function ρ satisfies $-\Delta_{p,g}\rho \geq 0$ in the sense of distributions.

In view of all these developments, it is natural to ask whether the Hardy and Leray inequalities given above can be combined under a single inequality for $\Omega \subset M$,

$$\int_\Omega |\nabla_g \phi|^p dv_g \geq H_p \int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p}|\phi|^p dv_g + L_p \int_\Omega \frac{|\phi(x)|^p}{\rho^p\left(\ln\frac{1}{\rho}\right)^p} dv_g \tag{3.7}$$

where H_p and L_p are positive constants.

While our aim is to answer above question, we see that our method not only combines Hardy and Leray inequalities but also allows us to achieve some other inequalities with explicit constants in the hyperbolic space $\mathbb{H}^n$.

We begin this section by proving a new L^p-Leray inequality with remainders. Indeed, our proof is simple and uses minimal assumption on the weight function ρ.

Theorem 3.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in M and $1 < p < \infty$. Let ρ be a nonnegative function on Ω such that $\sup_\Omega(\rho) < 1$. Then, the* *following inequality holds:*

$$\int_\Omega |\nabla_g\phi|^p dv_g \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla_g\rho|^p}{\rho^p(\log\frac{1}{\rho})^p}|\phi|^p dv_g + \left(\frac{p-1}{p}\right)^{p-1}\int_\Omega \frac{\Delta_{p,g}\rho}{(\rho\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g - \frac{(p-1)^p}{p^{p-1}}\int_\Omega \frac{|\nabla_g\rho|^p}{\rho^p\left(\log\frac{1}{\rho}\right)^{p-1}}|\phi|^p dv_g, \tag{3.8}$$

for all $\phi \in C_c^\infty(\Omega)$.

Proof. Let $u = -\log\left(\log\frac{1}{\rho}\right)$. A direct computation shows that

$$\Delta_{p,g}u = \frac{\Delta_{p,g}\rho}{(\rho\log\frac{1}{\rho})^{p-1}} - \frac{(p-1)|\nabla_g\rho|^p}{\rho^p\left(\log\frac{1}{\rho}\right)^{p-1}} + \frac{(p-1)|\nabla_g\rho|^p}{\rho^p\left(\log\frac{1}{\rho}\right)^p}, \quad \rho \neq 0, 1. \tag{3.9}$$

Multiplying both sides of (3.9) by the test function $|\phi|^p$ and integrating over Ω yields

$$\int_\Omega \Delta_{p,g}u|\phi|^p dv_g = \int_\Omega \frac{\Delta_{p,g}\rho}{(\rho\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g - (p-1)\int_\Omega \frac{|\nabla_g\rho|^p}{\rho^p(\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g + (p-1)\int_\Omega \frac{|\nabla_g\rho|^p}{\rho^p\left(\log\frac{1}{\rho}\right)^p}|\phi|^p dv_g. \tag{3.10}$$

Application of integration by parts (i.e., the divergence theorem) to the left side of (3.10) gives

$$\frac{1}{p}\int_\Omega \Delta_{p,g}u|\phi|^p dv_g = -\int_\Omega |\phi|^{p-1}\langle\nabla_g\phi, \nabla_g u\rangle|\nabla_g u|^{p-2}dv_g \leq \int_\Omega |\phi|^{p-1}|\nabla_g\phi||\nabla_g u|^{p-1}dv_g. \tag{3.11}$$

Recall Young's inequality with ϵ: for any $\epsilon > 0$, and $a, b \geq 0$,

$$ab \leq \epsilon a^p + (p\epsilon)^{-1/(p-1)}\frac{(p-1)}{p}b^{\frac{p}{p-1}}.$$

Hence by Young's inequality, we conclude that

$$\int_\Omega |\phi|^{p-1}|\nabla_g\phi||\nabla_g u|^{p-1}dv_g \leq \epsilon\int_\Omega |\nabla_g\phi|^p dv_g + c(\epsilon)\int_\Omega \frac{|\nabla_g\rho|^p}{\rho^p(\log\frac{1}{\rho})^p}|\phi|^p dv_g, \tag{3.12}$$

where $c(\epsilon) = \left(\frac{p-1}{p}\right)(p\epsilon)^{\frac{1}{1-p}}$. Substituting (3.12) into (3.10) gives

$$\begin{aligned}\int_\Omega |\nabla_g \phi|^p dv_g \geq \left(\frac{p-1}{p\epsilon}\right)\left(1-(p\epsilon)^{\frac{1}{1-p}}\right)\int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p(\log\frac{1}{\rho})^p}|\phi|^p dv_g \\ + \frac{1}{p\epsilon}\int_\Omega \frac{\Delta_{p,g}\rho}{(\rho\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g \\ - \frac{p-1}{p\epsilon}\int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p(\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g.\end{aligned} \tag{3.13}$$

Note that the function $\epsilon \to \frac{p-1}{p\epsilon}\left(1-(p\epsilon)^{\frac{1}{1-p}}\right)$ attains the maximum value for $\epsilon = \frac{p^{p-2}}{(p-1)^{p-1}}$, and this maximum is equal to $(\frac{p-1}{p})^p$. Now, we have the desired inequality (3.8),

$$\begin{aligned}\int_\Omega |\nabla_g \phi|^p dv_g \geq \left(\frac{p-1}{p}\right)^p\int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p(\log\frac{1}{\rho})^p}|\phi|^p dv_g \\ + \left(\frac{p-1}{p}\right)^{p-1}\int_\Omega \frac{\Delta_{p,g}\rho}{(\rho\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g \\ - \frac{(p-1)^p}{p^{p-1}}\int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p(\log\frac{1}{\rho})^{p-1}}|\phi|^p dv_g.\end{aligned}$$

□

Let us now turn our attention to the question of combining the Hardy and Leray type inequalities under a single inequality. We should note that the trace of such inequalities appears in [25]. Now, under new assumptions on the weight functions ρ and δ, our first result in this direction is as follows.

Theorem 3.2. *Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ and δ be nonnegative functions on M such that $\Delta_{p,g}\rho \geq C\frac{|\nabla_g \rho|^p}{\rho}$ and $-div_g(\rho^{p-C-1}\frac{|\nabla_g \delta|^{p-2}}{\delta^{p-2}}\nabla_g \delta) \geq 0$ in the sense of distributions where $C > 0$ and $p \geq 2$. Then, the following inequality holds:*

$$\begin{aligned}\int_M |\nabla_g \phi|^p dv_g \geq \left(\frac{C+1-p}{p}\right)^p\int_M \frac{|\nabla_g \rho|^p}{\rho^p}|\phi|^p dv_g \\ + \left(\frac{1}{2^{p-1}-1}\right)\frac{1}{p^p}\int_M \frac{|\nabla_g \delta|^p}{\delta^p}|\phi|^p dv_g\end{aligned} \tag{3.14}$$

for all $\phi \in C_c^\infty(M\backslash\rho^{-1}\{0\})$.

Proof. Let $\phi \in C_c^\infty(M\backslash\rho^{-1}\{0\})$ and define $\phi = \rho^\beta\psi$ where $\beta < 0$. Then, we have

$$|\nabla_g(\rho^\beta\psi)|^p = |\beta\rho^{\beta-1}\nabla_g\rho\psi + \rho^\beta\nabla_g\psi|^p.$$

We now use the following convexity inequality which is valid for any $a, b \in \mathbb{R}^n$ and $p \geq 2$,

$$|a+b|^p - |a|^p \geq p|a|^{p-2}a \cdot b + c(p)|b|^p$$

where $c(p) = \frac{1}{2^{p-1}-1}$. This yields

$$\begin{aligned} |\nabla_g \phi|^p \geq & |\beta|^p \rho^{p\beta-p}|\nabla_g \rho|^p |\psi|^p + \beta|\beta|^{p-2}\rho^{p\beta-p+1}\langle \nabla_g(|\psi|^p), \nabla_g \rho\rangle |\nabla_g \rho|^{p-2} \\ & + c(p)\rho^{p\beta}|\nabla_g \psi|^p. \end{aligned} \tag{3.15}$$

Applying integration by parts over M gives

$$\begin{aligned} \int_M |\nabla_g \phi|^p dv_g \geq |\beta|^p \int_M & \rho^{p\beta-p}|\nabla_g \rho|^p |\psi|^p dv_g \\ & - \beta|\beta|^{p-2}\int_M \mathrm{div}_g\left(\rho^{p\beta-p+1}|\nabla_g \rho|^{p-2}\nabla_g \rho\right)|\psi|^p dv_g \\ & + c(p)\int_M \rho^{p\beta}|\nabla_g \psi|^p dv_g. \end{aligned} \tag{3.16}$$

Since

$$\Delta_{p,g}\rho \geq \frac{C|\nabla_g \rho|^p}{\rho},$$

we obtain the following inequality

$$\mathrm{div}_g\left(\rho^{p\beta-p+1}|\nabla_g \rho|^{p-2}\nabla_g \rho\right) \geq (\beta p - p + 1 + C)\rho^{\beta p - p}|\nabla_g \rho|^p. \tag{3.17}$$

Substituting (3.17) in (3.16), we have

$$\begin{aligned} -\beta|\beta|^{p-2} & \int_M \mathrm{div}_g(\rho^{\beta p-p+1}\nabla_g \rho|\nabla_g \rho|^{p-2})|\psi|^p dv_g \\ & \geq -\beta|\beta|^{p-2}(\beta p - p + 1 + C)\int_M \rho^{\beta p-p}|\nabla_g \rho|^p |\psi|^p dv_g. \end{aligned} \tag{3.18}$$

Combining (3.18) and (3.16), we obtain

$$\int_M |\nabla_g \phi|^p dv_g \geq f(\beta)\int_M \rho^{\beta p-p}|\nabla_g \rho|^p |\psi|^p dv_g + c(p)\int_M \rho^{\beta p}|\nabla_g \psi|^p dv_g, \tag{3.19}$$

where

$$f(\beta) = -\beta|\beta|^{p-2}(\beta p - p + 1 + C) + |\beta|^p.$$

Since $f(\beta)$ attains its maximum for $\beta_0 = \frac{p-C-1}{p}$ and its maximum value is equal to $f(\beta_0) = \left(\frac{C+1-p}{p}\right)^p$. The inequality (3.19) becomes

$$\begin{aligned} \int_M |\nabla_g \phi|^p dv_g \geq & \left(\frac{C+1-p}{p}\right)^p \int_M \rho^{-C-1}|\nabla_g \rho|^p |\psi|^p dv_g \\ & + c(p)\int_M \rho^{p-C-1}|\nabla_g \psi|^p dv_g. \end{aligned} \tag{3.20}$$

Now, we focus on the second term on the right-hand side of this inequality. Let us define a new variable $\varphi(x) := \delta(x)^{-1/p}\psi(x)$ where $\delta(x)$ is a nonnegative function and $\delta(x) \in C^{\infty}(M\backslash\rho^{-1}\{0\})$. It follows from the convexity inequality that

$$|\nabla_g\psi|^p \geq \frac{|\varphi|^p}{p^p}\delta^{1-p}|\nabla_g\delta|^p + p^{1-p}|\nabla_g\delta|^{p-2}\delta^{2-p}\langle\nabla_g\delta, \nabla_g(|\varphi|^p)\rangle,$$

and therefore

$$\begin{aligned}\int_M \rho^{p-C-1}|\nabla_g\psi|^p dv_g \geq & \frac{1}{p^p}\int_M \rho^{p-C-1}|\nabla_g\delta|^p\delta^{1-p}|\varphi|^p dv_g \\ & + p^{1-p}\int_M \rho^{p-C-1}\frac{|\nabla_g\delta|^{p-2}}{\delta^{p-2}}\langle\nabla_g\delta, \nabla_g(|\varphi|^p)\rangle dv_g.\end{aligned} \tag{3.21}$$

Here, first we apply integration by parts to the second term on the right-hand side of (3.21) to get

$$\begin{aligned}\int_M \rho^{p-C-1}|\nabla_g\psi|^p dv_g \geq & \frac{1}{p^p}\int_M \rho^{p-C-1}|\nabla_g\delta|^p\delta^{1-p}|\varphi|^p dv_g \\ & - p^{1-p}\int_M \mathrm{div}_g(\rho^{p-C-1}\frac{|\nabla_g\delta|^{p-2}}{\delta^{p-2}}\nabla_g\delta)|\varphi|^p dv_g,\end{aligned}$$

then using the differential inequality $-\mathrm{div}_g(\rho^{p-C-1}\frac{|\nabla_g\delta|^{p-2}}{\delta^{p-2}}\nabla_g\delta) \geq 0$ and taking back substitution $\varphi = \rho^{\frac{-p+C+1}{p}}\phi\delta^{-1/p}$, we have

$$\int_M \rho^{p-C-1}|\nabla_g\psi|^p dv_g \geq \frac{1}{p^p}\int_M \frac{|\nabla_g\delta|^p}{\delta^p}|\phi|^p dv_g. \tag{3.22}$$

Substituting (3.22) into (3.20) gives the desired inequality

$$\int_M |\nabla_g\phi|^p dv_g \geq \left(\frac{C+1-p}{p}\right)^p \int_M \frac{|\nabla_g\rho|^p}{\rho^p}|\phi|^p dv_g + \frac{c(p)}{p^p}\int_M \frac{|\nabla_g\delta|^p}{\delta^p}|\phi|^p dv_g.$$

□

We note that a similar inequality also holds for $1 < p < 2$, and in this case, we use the following inequality

$$|a+b|^p - |a|^p \geq c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a\cdot b$$

where $c(p) > 0$ (see [29]).

A direct computation shows that $\delta := \ln\frac{1}{\rho}$ satisfies the assumption of Theorem 3.2. Hence, we have the following inequality which includes both Hardy- and Leray-type inequalities.

Corollary 3.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in M. Let ρ be a nonnegative function such that $\Delta_{p,g}\rho \geq \frac{C|\nabla_g \rho|^p}{\rho}$ in the sense of distributions where $C > 1$, $p \geq 2$ and $\sup_\Omega \rho < 1$. Then the following inequality holds,*

$$\begin{aligned}\int_\Omega |\nabla_g \phi|^p dv_g \geq \left(\frac{C+1-p}{p}\right)^p \int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p} |\phi|^p dv_g \\ + (\frac{1}{2^{p-1}-1}) \frac{1}{p^p} \int_\Omega \frac{|\nabla_g \rho|^p}{\rho^p \left(\ln \frac{1}{\rho}\right)^p} |\phi|^p dv_g\end{aligned} \tag{3.23}$$

for all $\phi \in C_c^\infty(\Omega \backslash \rho^{-1}\{0\})$.

The following theorem is the final theorem of this section, where we slightly change our assumptions on the weight functions ρ and δ.

Theorem 3.3. *Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ and δ be nonnegative functions on M such that $-\Delta_{p,g}\rho \geq 0$ and $-\mathrm{div}_g(\rho^{p-1}|\nabla_g \delta|^{p-2}\nabla_g \delta) \geq 0$ in the sense of distributions where $p \geq 2$. Then, we have*

$$\begin{aligned}\int_M |\nabla_g \phi|^p dv_g \geq & \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla_g \rho|^p}{\rho^p} |\phi|^p dv_g \\ & - \left(\frac{p-1}{p}\right)^{p-1} \int_M \frac{\Delta_{p,g}\rho}{\rho^{p-1}} |\phi|^p dv_g \\ & + (\frac{1}{2^{p-1}-1}) \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla_g \delta|^p}{\delta^p} |\phi|^p dv_g\end{aligned} \tag{3.24}$$

for all $\phi \in C_c^\infty(M \backslash \rho^{-1}\{0\})$.

Proof. The proof is similar to that of Theorem 3.2 and therefore omitted. □

In the following subsection, we present some applications of the above theorems in the hyperbolic space $\mathbb{H}^n$.

3.1. Hyperbolic space

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [15,33] for more precise information. The hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) is a complete simply connected Riemannian manifold having constant sectional curvature equal to -1. There are several models for $\mathbb{H}^n$ and we will use the Poincaré ball model $\mathbb{B}^n$ in this paper.

The Poincaré ball model for the hyperbolic space is

$$\mathbb{B}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x| < 1\}$$

endowed with the Riemannian metric $g_{\mathbb{B}^n} = p^2(x) g_{\mathbb{R}^n}$, where $p(x) = \frac{2}{1-|x|^2}$ and $g_{\mathbb{R}^n}$ is the canonical Euclidean metric. Hence, $\{p dx_i\}_{i=1}^n$ gives an orthonormal basis of the tangent space at $x = (x_1, \dots, x_n)$ in $\mathbb{B}^n$. The corresponding dual basis is $\{\frac{1}{p}\frac{\partial}{\partial x_i}\}_{i=1}^n$, thus the hyperbolic gradient and the Laplace Beltrami operator are

$$\nabla_{\mathbb{H}^n} u = \frac{\nabla u}{p},$$

$$\Delta_{\mathbb{H}^n} u = p^{-n}\mathrm{div}(p^{n-2}\nabla u);$$

where ∇ and div denote the Euclidean gradient and divergence in $\mathbb{R}^n$, respectively.

The hyperbolic distance $d_{\mathbb{H}^n}(x, y)$ between $x, y \in \mathbb{B}^n$ in the Poincaré ball model is given by the formula

$$d_{\mathbb{H}^n}(x, y) = \operatorname{arccosh}\Big(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big).$$

From this, we immediately obtain for $x \in \mathbb{B}^n$,

$$\begin{aligned}\rho(x) := d_{\mathbb{H}^n}(0, x) &= 2\operatorname{arctanh}|x| \\ &= \log\Big(\frac{1+|x|}{1-|x|}\Big)\end{aligned}$$

which is the distance from $x \in \mathbb{B}^n$ to the origin. Moreover, the geodesic curves passing through the origin are the diameters of $\mathbb{B}^n$ along with open arcs of circles in $\mathbb{B}^n$ perpendicular to the boundary at ∞, $\partial\mathbb{B}^n = \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

The hyperbolic volume element is given by

$$dv = p^n(x)dx = \Big(\frac{2}{1-r^2}\Big)^n r^{n-1}drd\sigma$$

where dx denotes the Lebesgue measure in $\mathbb{B}^n$ and $d\sigma$ is the normalized surface measure on $\mathbb{S}^{n-1}$. Note that the hyperbolic volume element dv can be written in geodesic polar coordinates as

$$dv = (\sinh\rho)^{n-1}d\rho d\sigma.$$

A hyperbolic ball in $\mathbf{B^n}$ with center 0 and hyperbolic radius $R \in (0, \infty)$ is defined by

$$\mathbf{B}_R = \{x \in \mathbb{H}^n \mid d_{\mathbb{H}^n}(0, x) < R\};$$

and note that $\mathbf{B}_R$ is also the Euclidean ball with center 0 and radius $S = \tanh\frac{R}{2} \in (0, 1)$.

The following polar coordinate integration formula holds for $f \in L^1(\mathbb{H}^n)$,

$$\int_{\mathbb{H}^n} f\,dv = \int_0^\infty \Big(\int_{\mathbb{S}^{n-1}} f(\rho, \theta)d\sigma\Big)(\sinh\,\rho)^{n-1}d\rho.$$

It is clear that the volume of a hyperbolic ball $\mathbf{B}_R$ is

$$\mathrm{Vol}(\mathbf{B}_R) = n\omega_n \int_0^R (\sinh\rho)^{n-1} d\rho,$$

where ω_n denotes the volume of the n-dimensional Euclidean unit ball.

Furthermore, the following estimate holds,

$$\mathrm{Vol}(\mathbf{B}_R) = n\omega_n \int_0^R (\sinh\rho)^{n-1} d\rho \le \omega_n (\sinh R)^n.$$

We have the following two relations for the distance function $d(x) := \log(\frac{1+|x|}{1-|x|})$

$$\begin{aligned} |\nabla_{\mathbb{H}^n} d| &= 1, \\ \Delta_{\mathbb{H}^n} d &\ge \frac{n-1}{\rho}, \quad x \neq 0. \end{aligned}$$

The following Poincaré inequality in $\mathbb{H}^n$ is due to McKean [30] for $p = 2$ and generalized for any $p > 1$ by Strichartz [35],

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^p dv \ge \left(\frac{n-1}{p}\right)^p \int_{\mathbb{H}^n} |\phi|^p dv \tag{3.25}$$

where $\phi \in C_c^\infty(\mathbb{H}^n)$ and $1 \le p < \infty$. Furthermore, the constant $(\frac{n-1}{p})^p$ is sharp (see [6,31]).

3.2. Applications of Theorems 3.1–3.3

In this subsection, we present several Hardy-, Leray- and Poincaré-type inequalities with remainders after making suitable choices of weights ρ and δ satisfying the hypothesis in Theorems 3.1–3.3 in the hyperbolic space $\mathbb{H}^n$.

Using Theorem 3.1 with $\rho := d_{\mathbb{H}^n}(0, x) = \log(\frac{1+|x|}{1-|x|})$, we immediately obtain the following corollary.

Corollary 3.2. *Let $\mathbf{B}_1$ be the unit hyperbolic ball in $\mathbb{H}^n$, $n \ge 2$ and $1 < p \le n$. Then we have,*

$$\begin{aligned} \int_{\mathbf{B}_1} |\nabla_{\mathbb{H}^n}\phi|^p dv \ge& \left(\frac{p-1}{p}\right)^p \int_{\mathbf{B}_1} \frac{|\phi|^p}{\rho^p (\log\frac{1}{\rho})^p} dv \\ &+ (n-p)\left(\frac{p-1}{p}\right)^{p-1} \int_{\mathbf{B}_1} \frac{|\phi|^p}{\rho^p (\log\frac{1}{\rho})^{p-1}} dv \end{aligned} \tag{3.26}$$

for all $\phi \in C_c^\infty(\mathbf{B}_1)$. Furthermore, the constant $(\frac{p-1}{p})^p$ is sharp.

Proof. To show that the constant $(\frac{p-1}{p})^p$ is sharp, we use the following function:

$$\phi(\rho) = \begin{cases} \frac{\rho}{a}\left(\ln\frac{1}{a}\right)^{\frac{p-1}{p}(1+\epsilon)} & \text{if} \quad 0 \le \rho \le a \\ \left(\ln\frac{1}{\rho}\right)^{\frac{p-1}{p}(1+\epsilon)} & \text{if} \quad a \le \rho \le 1 \end{cases}$$

where a is a constant. Notice that $\phi(\rho)$ is a smooth function with compact support in $\mathbb{H}^n$, direct and tedious computation shows that $(\frac{p-1}{p})^p$ is the best constant,

$$\left(\frac{p-1}{p}\right)^p = \lim_{\epsilon\to 0} \frac{\int_{\mathbf{B}_1} |\nabla_{\mathbb{H}^n}\phi|^p dv}{\int_{\mathbf{B}_1} \frac{|\phi|^p}{\rho^p(\log\frac{1}{\rho})^p} dv}.$$

□

We now turn our attention to some applications of Theorem 3.2. First, let us consider the pair of functions,

$$\rho := \log\left(\frac{1+|x|}{1-|x|}\right) \quad \text{and} \quad \delta := e^{(2^{p-1}-1)^{\frac{1}{p}}(1-n)\rho}.$$

It is obvious that they fulfill the hypotheses of Theorem 3.2, hence we have the following Hardy–Poincaré-type inequality.

Corollary 3.3. *Let $\mathbf{B}_R$ be the hyperbolic ball in $\mathbb{H}^n$, centered at 0 and radius $R = \frac{p-1}{n-1}(\frac{1}{2^{p-1}-1})^{\frac{1}{p}}$ where $2 \le p \le n$. Then, the following inequality holds,*

$$\int_{\mathbf{B}_R} |\nabla_{\mathbb{H}^n}\phi|^p dv \ge \left(\frac{n-p}{p}\right)^p \int_{\mathbf{B}_R} \frac{|\phi|^p}{\rho^p} dv + \left(\frac{n-1}{p}\right)^p \int_{\mathbf{B}_R} |\phi|^p dv \tag{3.27}$$

for all $\phi \in C_c^\infty(\mathbf{B}_R)$.

On the other hand, by making the choices

$$\rho := \log\left(\frac{1+|x|}{1-|x|}\right) \quad \text{and} \quad \delta := \left(\ln\frac{1}{\rho}\right)^{p-1},$$

we derive the following Hardy–Leray-type inequality.

Corollary 3.4. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $2 \le p \le n$ and $\sup_\Omega d < 1$. Then for all $\phi \in C_c^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv \ge &\left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{\rho^p} dv \\ &+ \left(\frac{1}{2^{p-1}-1}\right)\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\phi|^p}{\rho^p\left(\ln\frac{1}{\rho}\right)^p} dv. \end{aligned} \tag{3.28}$$

Finally, let us look at the some consequences of Theorem 3.3. We consider various functions ρ and δ to achieve different inequalities. Although some of the following inequalities are similar to each other, we should note that they contain different constants. To avoid confusion, we always first write down the term containing the sharp constant and the name associated with that term.

In the following corollaries (Corollaries 3.5–3.8), $d(x) = \log(\frac{1+|x|}{1-|x|})$ denotes the hyperbolic distance between the origin and the point $x \in \mathbb{B}^n$.

Now, let us consider the special functions

$$\rho := d^{\frac{p-n}{p-1}} \quad \text{and} \quad \delta := \ln \frac{1}{d}.$$

We have the Hardy–Leray-type inequality.

Corollary 3.5. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $2 \leq p \leq n$ and $\sup_\Omega d < 1$. Then for all $\phi \in C_c^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dv \\ + \left(\frac{1}{2^{p-1}-1}\right)\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p\left(\ln\frac{1}{d}\right)^p} dv. \end{aligned} \tag{3.29}$$

Another application of Theorem 3.3 with special functions

$$\rho := d^{\frac{p-n}{p-1}} \quad \text{and} \quad \delta := e^{-d}$$

leads the following Hardy-Poincaré type inequality.

Corollary 3.6. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $2 \leq p \leq n$ and $\sup_\Omega d < 1$. Then for all $\phi \in C_c^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dv \\ + \left(\frac{1}{2^{p-1}-1}\right)\left(\frac{p-1}{p}\right)^p \int_\Omega |\phi|^p dv. \end{aligned} \tag{3.30}$$

On the other hand, by making the choices

$$\rho := \ln \frac{1}{d} \quad \text{and} \quad \delta := e^{-d}$$

we derive the following Leray–Poincaré inequality.

Corollary 3.7. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $2 \leq p \leq n$ and $\sup_\Omega d < e^{\frac{1-p}{n-p}}$. Then for all $\phi \in C_c^\infty(\Omega)$, we have*

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p(\ln\frac{1}{d})^p} dv \\ + \left(\frac{1}{2^{p-1}-1}\right)\left(\frac{p-1}{p}\right)^p \int_\Omega |\phi|^p dv. \end{aligned} \tag{3.31}$$

Another consequence of the Theorem 3.3 with the special functions

$$\rho := e^{\frac{(1-n)d}{p-1}} \quad \text{and} \quad \delta := \ln \frac{1}{d}$$

leads us to the following Poincaré–Leray-type inequality.

Corollary 3.8. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $2 \le p \le n$ and $\sup_\Omega d < \frac{n-p}{n-1}$. Then, we have*

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv \ge \left(\frac{n-1}{p}\right)^p \int_\Omega |\phi|^p dv \\ + \left(\frac{1}{2^{p-1}-1}\right)\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p(\ln\frac{1}{d})^p} dv \end{aligned} \tag{3.32}$$

for all $\phi \in C_c^\infty(\Omega)$.

3.3. Applications of Theorem 2.1

We are now ready to present some applications of Theorem 2.1.

In our first result below, we consider the Hardy potential.

Corollary 3.9. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $n \ge 2$, $0 \in \Omega$, $H_p = (\frac{n-p}{p})^p$ and $V(\rho) = \frac{c}{\rho^p}$ where $\rho = \log(\frac{1+|x|}{1-|x|})$. Then, the problem (1.1) has no generalized positive local solution off of $\mathcal{K} = \{0\}$ if $c > H_p$ and $\frac{2n}{n+1} \le p < 2$.*

Proof. Given $\epsilon > 0$, we define the radial function $\phi(x) = \varphi(\rho)$ by

$$\varphi(\rho) = \begin{cases} \epsilon^{\frac{-n}{p}}\rho & \text{if } \; 0 \le \rho \le \epsilon, \\ \rho^{-\frac{n-p}{p}} & \text{if } \; \epsilon \le \rho \le 1, \\ 2-\rho & \text{if } \; 1 \le \rho \le 2, \\ 0 & \text{if } \; \rho \ge 2. \end{cases} \tag{3.33}$$

Then,

$$|\nabla_{\mathbb{H}^n}\phi|^p = \begin{cases} \epsilon^{-n} & \text{if } \; 0 < \rho < \epsilon, \\ (\frac{n-p}{p})^p\rho^{-n} & \text{if } \; \epsilon < \rho < 1, \\ 1 & \text{if } \; 1 < \rho < 2, \\ 0 & \text{if } \; \rho > 2. \end{cases} \tag{3.34}$$

Without loss of generality, we assume that $\mathbf{B}_2 = \{x \in \mathbb{H}^n \mid \rho < 2\} \subset \Omega$; if not, we simply redefine ϕ, replacing 2 by R where $\mathbf{B}_R \subset \Omega$. This results only in notational changes in the proof that follows.

We want to show that

$$\sigma_{inf} := \inf_{0 \not\equiv \phi \in C_c^\infty(\Omega\setminus\mathcal{K})} \frac{\int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv - \int_\Omega \frac{(1-\epsilon)c}{\rho^p}|\phi|^p(x) dv}{\int_\Omega |\phi|^p dv} = -\infty.$$

Direct computation shows that

$$\int_{\Omega} |\nabla_{\mathbb{H}^n} \phi|^p dv = n\omega_n \int_0^{\epsilon} \epsilon^{-n} (\sinh \rho)^{n-1} d\rho + n\omega_n \int_{\epsilon}^{1} \left(\frac{n-p}{p}\right)^p \rho^{-n} (\sinh \rho)^{n-1} d\rho + n\omega_n \int_1^2 (\sinh \rho)^{n-1} d\rho. \tag{3.35}$$

Similarly, we get

$$(1-\epsilon)c \int_{\Omega} \frac{|\phi|^p}{\rho^p} dv = c(1-\epsilon) n\omega_n \Big(\int_0^{\epsilon} \epsilon^{-n} (\sinh \rho)^{n-1} d\rho + \int_{\epsilon}^{1} \rho^{-n} (\sinh \rho)^{n-1} d\rho + \int_1^2 \left(\frac{2-\rho}{\rho}\right)^p (\sinh \rho)^{n-1} d\rho \Big). \tag{3.36}$$

It is clear that

$$\int_{\Omega} |\nabla_{\mathbb{H}^n} \phi|^p dv - \int_{\Omega} \frac{(1-\epsilon)c}{\rho^p} |\phi|^p dv = n\omega_n A \int_0^{\epsilon} (\sinh \rho)^{n-1} d\rho + n\omega_n B \int_{\epsilon}^{1} \rho^{-n} (\sinh \rho)^{n-1} d\rho + n\omega_n \int_1^2 (\sinh \rho)^{n-1} d\rho + n\omega_n C \int_1^2 \left(\frac{2-\rho}{\rho}\right)^p (\sinh \rho)^{n-1} d\rho, \tag{3.37}$$

where $A = \epsilon^{-n}\big(1 - c(1-\epsilon)\big)$, $B = (\frac{n-p}{p})^p - c(1-\epsilon)$ and $C = -c(1-\epsilon)$. Note that the first, third and fourth integrals on the right hand side of (3.37) have finite values. Hence, we have

$$\int_{\Omega} |\nabla_{\mathbb{H}^n} \phi|^p dv - \int_{\Omega} \frac{(1-\epsilon)c}{\rho^p} |\phi|^p dv = n\omega_n B \int_{\epsilon}^{1} \rho^{-n} (\sinh \rho)^{n-1} d\rho + \tilde{C} \tag{3.38}$$

where $\tilde{C}$ is a positive constant.

On the other hand,

$$\begin{aligned}\int_{\Omega} |\phi|^p dv &= \int_{\mathbf{B}_\epsilon} \epsilon^{-n} \rho^p dv + \int_{\mathbf{B}_1 \setminus \mathbf{B}_\epsilon} \rho^{p-n} dv + \int_{\mathbf{B}_2 \setminus \mathbf{B}_1} (2-\rho)^p dv \\ &\geq n\omega_n \int_{\epsilon}^{1} \rho^{p-n} (\sinh \rho)^{n-1} d\rho \\ &\geq n\omega_n \int_{\epsilon}^{1} \rho^{p-1} d\rho \\ &= n\omega_n \frac{1-\epsilon^p}{p}\end{aligned} \tag{3.39}$$

since $\sinh\rho \geq \rho$. Substituting (3.38) and (3.39) into the Rayleigh quotient gives

$$\begin{aligned}\mathcal{R} &= \frac{\int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv - \int_\Omega \frac{c(1-\epsilon)}{\rho^p}|\phi|^p dv}{\int_\Omega \phi^p dv} \\ &\leq \frac{n\omega_n\Big(\Big(\Big(\frac{n-p}{p}\Big)^p - c(1-\epsilon)\Big)\int_\epsilon^1 \rho^{-n}(\sinh\rho)^{n-1}d\rho\Big) + \tilde{C}}{n\omega_n\frac{1-\epsilon^p}{p}}.\end{aligned} \tag{3.40}$$

Note that $(\frac{n-p}{p})^p - c(1-\epsilon) < 0$ and using the fact $\rho \leq \sinh\rho \leq 2\rho$ for $\rho \in [0, 1]$, we get

$$\lim_{\epsilon\to 0}\int_\epsilon^1 \rho^{-n}(\sinh\rho)^{n-1}d\rho = +\infty.$$

Hence, for $\epsilon > 0$ small enough,

$$\inf_{0\not\equiv\phi\in C_c^\infty(\Omega\backslash\{0\})}\frac{\int_\Omega |\nabla_{\mathbb{H}^n}\phi|^p dv - \int_\Omega \frac{(1-\epsilon)c}{\rho^p}|\phi|^p dv}{\int_\Omega |\phi|^p dv} = -\infty.$$

The proof of Corollary 3.9 is now complete. □

Corollary 3.10. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$, $n \geq 2$, $0 \in \Omega$, $H_p = (\frac{n-p}{p})^p$, $V(\rho) = \frac{c}{\rho^p} + \frac{\beta}{\rho^p}\sin(\frac{1}{\rho^\alpha})$, where $\rho = \log(\frac{1+|x|}{1-|x|})$, $c > 0$, $\alpha > 0$ and $\beta \in \mathbb{R}^n\backslash\{0\}$. Then the problem (1.1) has no generalized positive local solution off of $\mathcal{K}$ if $c > H_p$ and $\frac{2n}{n+1} \leq p < 2$.*

Proof. In order to show that $\sigma_{\inf} = -\infty$, we use the same test function ϕ in Corollary 3.9. The details are omitted. □

In the following corollary, we use the Leray potential obtained from Corollary 3.2 which is a result of Theorem 3.1.

Corollary 3.11. *Let $\mathbf{B}_1$ be the unit hyperbolic ball in $\mathbb{H}^n$, $n \geq 2$, $V(\rho) = \frac{c}{\rho^p(\ln\frac{1}{\rho})^p}$ where $\rho = \log(\frac{1+|x|}{1-|x|})$ and $L_p = (\frac{p-1}{p})^p$. Then, (1.1) has no generalized positive local solution off of $\mathcal{K} = \{0\} \cup \partial\mathbf{B}_1$ if $c > L_p$ and $\frac{2n}{n+1} \leq p < 2$.*

Proof. Let $\phi(x) = \varphi(\rho)$ be the radial function defined by

$$\varphi(\rho) = \begin{cases} 1 & \text{if } \ 0 \leq \rho \leq \frac{1}{e} \\ \Big(\ln\frac{1}{\rho}\Big)^{(\frac{p-1}{p})(1+\epsilon)} & \text{if } \ \frac{1}{e} \leq \rho \leq 1, \end{cases} \tag{3.41}$$

where $\epsilon > 0$. Then

$$|\nabla_{\mathbb{H}^n}\phi(x)|^p = \begin{cases} 0 & \text{if } \ 0 < \rho < \frac{1}{e}, \\ \Big(\frac{p-1}{p}\Big)^p(1+\epsilon)^p(\ln\frac{1}{\rho})^{p\epsilon-1-\epsilon}\frac{1}{\rho^p} & \text{if } \ \frac{1}{e} < \rho < 1. \end{cases} \tag{3.42}$$

A direct computation shows that

$$\int_{\mathbf{B}_1} |\nabla_{\mathbb{H}^n}\phi|^p dv = nw_n \left(\frac{p-1}{p}\right)^p (1+\epsilon)^p \int_{\frac{1}{e}}^{1} \left(\ln\frac{1}{\rho}\right)^{p\epsilon-1-\epsilon} \frac{1}{\rho^p}(\sinh\rho)^{n-1}d\rho. \tag{3.43}$$

Similarly, we get

$$\begin{aligned}\int_{\mathbf{B}_1} \frac{c}{\rho^p\left(\ln\frac{1}{\rho}\right)^p}|\phi|^p dv =& nw_n c\int_0^{\frac{1}{e}} \left(\ln\frac{1}{\rho}\right)^{-p} \frac{1}{\rho^p}(\sinh\rho)^{n-1}d\rho \\ &+ nw_n c\int_{\frac{1}{e}}^{1} \left(\ln\frac{1}{\rho}\right)^{p\epsilon-1-\epsilon} \frac{1}{\rho^p}(\sinh\rho)^{n-1}d\rho.\end{aligned} \tag{3.44}$$

Since the first integral on the right-hand side of (3.44) is finite, we can write

$$\int_{\mathbf{B}_1} \frac{c}{\rho^p(\ln\frac{1}{\rho})^p}|\phi|^p dv = nw_n c\int_{\frac{1}{e}}^{1} \left(\ln\frac{1}{\rho}\right)^{p\epsilon-1-\epsilon} \frac{1}{\rho^p}(\sinh\rho)^{n-1}d\rho + C_1 \tag{3.45}$$

where $C_1 > 0$.

On the other hand,

$$\begin{aligned}\int_{\mathbf{B}_1} |\phi|^p dv &= \int_{\mathbf{B}_{\frac{1}{e}}} dv + \int_{\mathbf{B}_1\setminus\mathbf{B}_{\frac{1}{e}}} (\ln\frac{1}{\rho})^{(p-1)(1+\epsilon)}dv \\ &\geq \mathrm{Vol}(\mathbf{B}_{\frac{1}{e}}) = C_2 > 0.\end{aligned} \tag{3.46}$$

Substituting (3.43)–(3.46) into the Rayleigh quotient gives

$$\begin{aligned}\mathcal{R} &= \frac{\int_{\mathbf{B}_1} |\nabla_{\mathbb{H}^n}\phi|^p dv - \int_{\mathbf{B}_1} \frac{(1-\epsilon)c}{\rho^p(\ln\frac{1}{\rho})^p}|\phi|^p dv}{\int_{\mathbf{B}_1} |\phi|^p dv} \\ &\leq \frac{n\omega_n\left((\frac{p-1}{p})^p(1+\epsilon)^p - c + c\epsilon\right)\int_{\frac{1}{e}}^{1}(\ln\frac{1}{\rho})^{p\epsilon-1-\epsilon}\frac{1}{\rho^p}(\sinh\rho)^{n-1}d\rho + C_1}{C_2}.\end{aligned} \tag{3.47}$$

The conclusion

$$\lim_{\epsilon\to 0}\mathcal{R} = -\infty$$

follows from

$$\lim_{\epsilon\to 0}\int_{\frac{1}{e}}^{1} \left(\ln\frac{1}{\rho}\right)^{p\epsilon-1-\epsilon} \rho^{n-p-1}d\rho = +\infty$$

since $\rho \leq \sinh\rho \leq 2\rho$ for $\rho \in [0,1]$ and $(\frac{p-1}{p})^p(1+\epsilon)^p - c + c\epsilon < 0$ for $\epsilon > 0$ small enough.

The substitution $z = \ln(\dfrac{1}{\rho})$ gives

$$I := \int_{\frac{1}{e}}^{1} \left(\ln \frac{1}{\rho}\right)^{p\epsilon-1-\epsilon} \rho^{n-p-1} d\rho = \int_0^1 z^{p\epsilon-1-\epsilon} e^{(p-n)z} dz.$$

Thus,

$$I \geq \frac{e^{p-n}}{p\epsilon - \epsilon} \to \infty$$

as $\epsilon \to 0$, which completes the proof. $\square$

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Accepted: 18 March 2021